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Year 12 Mathematics Specialist 2018

Test Number 3: Vectors

Resource Rich

Name: Solutions **Teacher: DDA**

Marks: **45**

Time Allowed: **45 minutes**

Instructions: You are permitted 1 A4 page of notes and your calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 1 **[1, 2, 3 = 6 marks]**

If $\boldsymbol{a} = < -2,3,1 >$ and $\boldsymbol{b} = < 3,1,-5 >$ find:

a) $-\boldsymbol{a} - 5\boldsymbol{b}$

$$-\begin{bmatrix}-2\\3\\1\end{bmatrix}-5\begin{bmatrix}3\\1\\-5\end{bmatrix}$$

$$\begin{bmatrix}-13\\-8\\24\end{bmatrix}$$

b) The size of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\approx 111.2°$

$$\text{angle}\left(\begin{bmatrix}-2\\3\\1\end{bmatrix},\begin{bmatrix}3\\1\\-5\end{bmatrix}\right)$$

111.186394

c) The acute angle between $\boldsymbol{a}$ and the $x - y$ plane.

Angle required is the angle between <-2,3,1> and <-2,3,0>

$\approx 15.5°$

$$\text{angle}\left(\begin{bmatrix}-2\\3\\1\end{bmatrix},\begin{bmatrix}-2\\3\\0\end{bmatrix}\right)$$

15.50135957

or

Angle plane−line

Angle between line **r=a+λb** and plane **r.n**=k

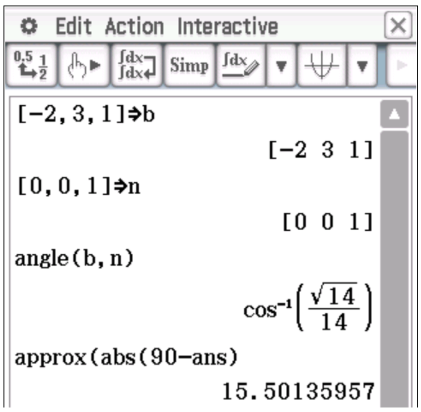
Normal to plane is <0,0,1>

Need the complement of the angle between <-2, 3,1> and <0,0,1>

$\approx 15.5°$

$$90-\text{angle}\left(\begin{bmatrix}-2\\3\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

15.50135957



Question 2 **[2 mark]**

Find the vector equation of the line perpendicular to the plane $2x+3y-z=5$ and that contains the point $P(1,-2,0)$.

Normal to the plane is <2,3,-1>. Therefore, the line is parallel to <2,3,-1>. ✓

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$ ✓

Question 3 **[2 marks]**

Find the vector equation of a plane that contains the line $\mathbf{r}(t) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and the point $P(-1,2,-4)$.

A second vector on the plane, not parallel to <3,0,2>:

<-1,2,-4> - <1,2,-1> = <-2,0,-3> or <2,0,3> ✓

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$ ✓

Question 4 **[1,1,3,2 = 7 marks]**

Two parallel planes have the following equations

Plane Π: $\quad \boldsymbol{r}.\begin{pmatrix}2\\-3\\6\end{pmatrix} = 14$ $\qquad$ Plane Ω: $\quad \boldsymbol{r}.\begin{pmatrix}2\\-3\\6\end{pmatrix} = 42.$.

a) Point A with position vector $4\boldsymbol{i} + 2\boldsymbol{j} + c\boldsymbol{k}$ lies on the plane Π. Find the value of c.

$c=2$ ✓

$$\text{solve}\left(\text{dotP}\left(\begin{bmatrix}4\\2\\x\end{bmatrix},\begin{bmatrix}2\\-3\\6\end{bmatrix}\right)=14\right)$$

$\{x=2\}$

b) Determine the equation of the line L that passes through A and is perpendicular to plane Π.

$$r = \begin{pmatrix}4\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$$ ✓

c) Determine the position vector of B, the point of intersection of line L with plane Ω.

$$\begin{pmatrix}4+2\lambda\\2-3\lambda\\2+6\lambda\end{pmatrix}.\begin{pmatrix}2\\-3\\6\end{pmatrix} = 42$$ ✓

$$\text{solve}\left(\text{dotP}\left(\begin{bmatrix}4+2x\\2-3x\\2+6x\end{bmatrix},\begin{bmatrix}2\\-3\\6\end{bmatrix}\right)=42\right)$$

$\left\{x=\frac{4}{7}\right\}$

$\lambda = \frac{4}{7}$ ✓

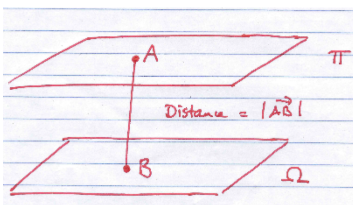
$$\begin{bmatrix}4+2x\\2-3x\\2+6x\end{bmatrix}\Big| x=\frac{4}{7}$$

$$\begin{bmatrix}\frac{36}{7}\\\frac{2}{7}\\\frac{38}{7}\end{bmatrix}$$

B is at $\begin{pmatrix}\frac{36}{7}\\\frac{2}{7}\\\frac{38}{7}\end{pmatrix}$ ✓

d) Determine the exact distance between the planes Π and Ω.

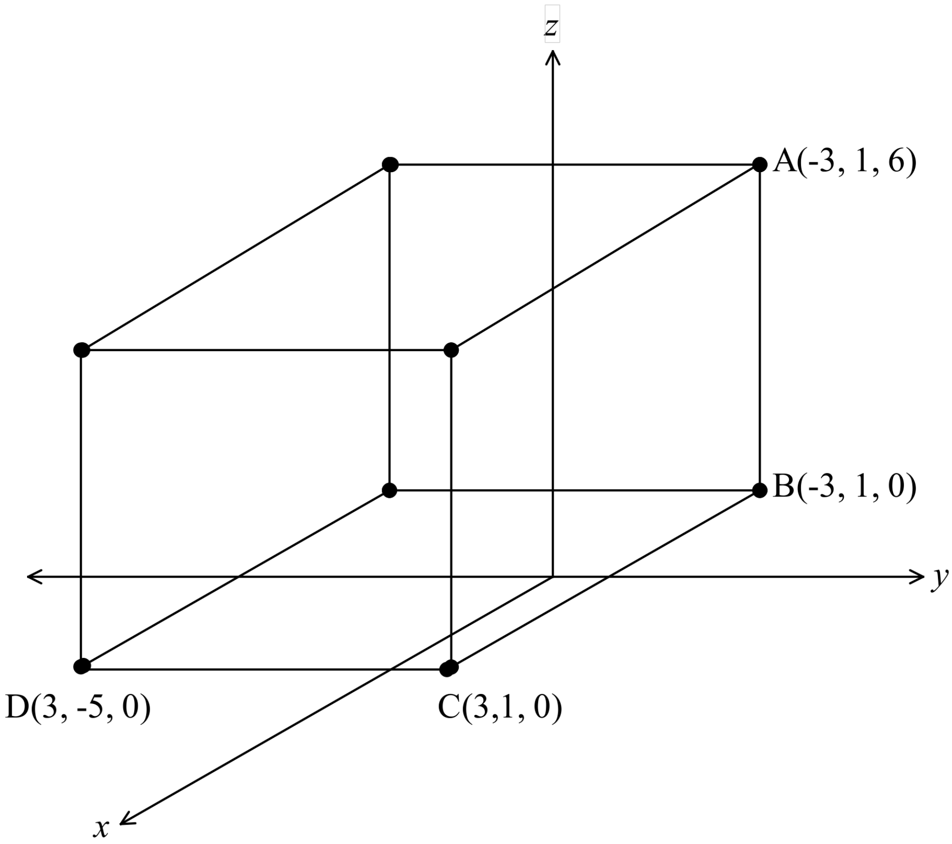
Distance = $|\overrightarrow{AB}| = 4\ units$ ✓✓



$$\text{norm}\left(\begin{bmatrix}4\\2\\2\end{bmatrix}-\begin{bmatrix}\frac{36}{7}\\\frac{2}{7}\\\frac{38}{7}\end{bmatrix}\right)$$

Question 5 **[3 marks]**

Find the equation of a sphere that fits exactly inside the cube on the diagram below.



Centre = <0,-2,3>

Radius = 3

Equation: $\left|\boldsymbol{r} - \begin{pmatrix}0\\-2\\3\end{pmatrix}\right| = 3$ or $x^2 + (y+2)^2 + (z-3)^2 = 9$

Question 6 **[1,1, 3 = 5 marks]**

A little boy, holding a sandwich in his hand at (0, 0, 0.5), is running along the street such that the position vector of the sandwich is $\mathbf{r}(t)=\begin{pmatrix}0\\0\\0.5\end{pmatrix}+t\begin{pmatrix}0.5\\0.5\\0\end{pmatrix}$ where t is measured in seconds from $t=0$.

A kookaburra at $(-5.5,-1.5,\,4.5)$ eyed off the sandwich for one second then swooped down with a velocity of $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ to pinch the sandwich.

(a) Show that the position vector of the kookaburra from $t=1$ is $\mathbf{r}_k(t)=\begin{pmatrix}-7.5\\-2.5\\5.5\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}$.

$$\mathbf{r}_k(t)=\begin{pmatrix}-5.5\\-1.5\\4.5\end{pmatrix}+(t-1)\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\mathbf{r}_k(t)=\begin{pmatrix}-5.5\\-1.5\\4.5\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}-1\times\begin{pmatrix}2\\1\\-1\end{pmatrix}$$ ✓

$$\mathbf{r}_k(t)=\begin{pmatrix}-7.5\\-2.5\\5.5\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

(b) How fast did the kookaburra fly? Distances are measured in metres.

$$\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right| = \sqrt{4+1+1} = \sqrt{6}\ m/s \approx 2.45 m/s$$ ✓

(c) How many seconds does the kookaburra take to steal the sandwich (not including the second when the bird is eyeing off the sandwich).

$${}_B\underset{\sim}{r}_K = \begin{pmatrix}15/2\\15/2\\-5\end{pmatrix} \qquad {}_B\underset{\sim}{v}_K = \begin{pmatrix}-3/2\\-1/2\\1\end{pmatrix}$$

$${}_B\underset{\sim}{v}_K \cdot \left({}_B\underset{\sim}{r}_K + t\,{}_B\underset{\sim}{v}_K\right) = 0$$

$$\begin{pmatrix}-3/2\\-1/2\\1\end{pmatrix} \cdot \begin{pmatrix}\frac{15}{2}-\frac{3t}{2}\\ \frac{15}{2}-\frac{1}{2}t\\ -5+t\end{pmatrix} = 0 \Rightarrow t = 5.$$ ✓ ✓

Closest App Dot

[0,0,0.5]⇒Ra

$$\begin{bmatrix}0 & 0 & \frac{1}{2}\end{bmatrix}$$

[0.5,0.5,0]⇒Va

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}$$

[-7.5,-2.5,5.5]⇒Rb

$$\begin{bmatrix}-\frac{15}{2} & -\frac{5}{2} & \frac{11}{2}\end{bmatrix}$$

[2,1,-1]⇒Vb

$$\begin{bmatrix}2 & 1 & -1\end{bmatrix}$$

Ra−Rb⇒R

$$\begin{bmatrix}\frac{15}{2} & \frac{5}{2} & -5\end{bmatrix}$$

Va−Vb⇒V

$$\begin{bmatrix}-\frac{3}{2} & -\frac{1}{2} & 1\end{bmatrix}$$

dotP(V,t×V+R)=0

$$\frac{\frac{t}{2}-\frac{5}{2}}{2}+\frac{3\cdot\left(\frac{3\cdot t}{2}-\frac{15}{2}\right)}{2}+t-5=0$$

simplify(ans)

$$\frac{7\cdot(t-5)}{2}=0$$

solve(ans,t)⇒soln

{t=5}

The kookaburra takes 4 seconds to steal the sandwich. (He waits for one second). ✓

Or

$$\begin{pmatrix}0\\0\\0.5\end{pmatrix} + t\begin{pmatrix}0.5\\0.5\\0\end{pmatrix} = \begin{pmatrix}-7.5\\-2.5\\5.5\end{pmatrix} + t\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$z: 0.5 = 5.5 - t \Rightarrow t = 5$ ✓

Check:

$y: 0.5t = -2.5 + t$

$2.5 = 0.5t \Rightarrow t = 5$

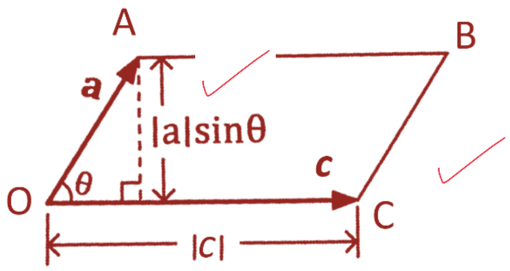
$x: 0.5t = -7.5 + 2t$

$7.5 = 1.5t \Rightarrow t = 5$ ✓

Question 7 **[3, 2 = 5 marks]**

(a) OABC is a parallelogram with OA parallel to CB. Let $\overrightarrow{\boldsymbol{OA}} = \boldsymbol{a}$ and $\overrightarrow{\boldsymbol{OC}} = \boldsymbol{c}$.

Prove that the area of the parallelogram OABC is $|\boldsymbol{a}\times\boldsymbol{c}|$.

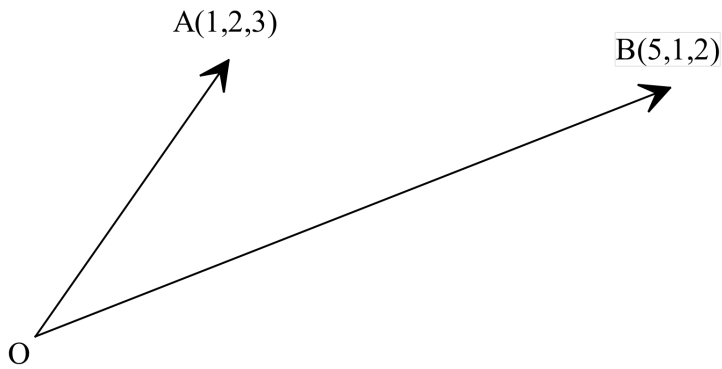


$A = b\times h$

$= |\boldsymbol{c}|\times|\boldsymbol{a}|\sin\theta$

$= |\boldsymbol{a}\times\boldsymbol{c}|$

(b) Hence, use vectors methods to determine the area of the triangle AOB in the diagram below.



crossP($\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5\\1\\2\end{bmatrix}$

$\begin{bmatrix}1\\13\\-9\end{bmatrix}$

norm(

$\sqrt{251}$

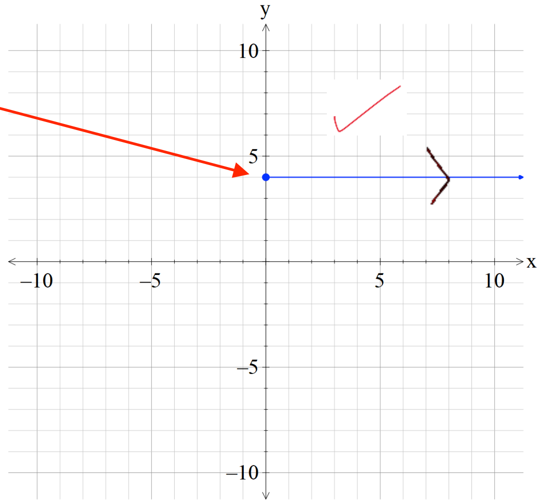
Area = $\frac{\sqrt{251}}{2}\ units^2$

Question 8 **[2, 2, 3 = 7 marks]**

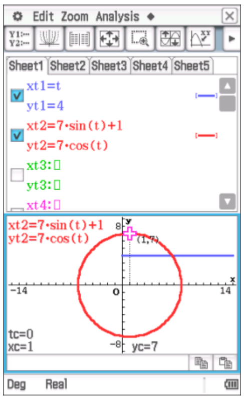
Find the Cartesian equation of the path traced by the point P with position vector $\boldsymbol{r}(t)$, where t represents time. Sketch the path, indicating starting position and the direction of motion.

a) $\boldsymbol{r}(t) = \begin{pmatrix} t \\ 4 \end{pmatrix}$

Starting position

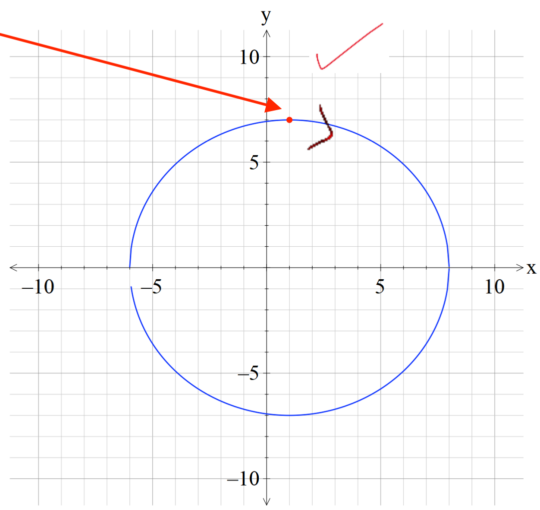


$y = 4 : x \geq 0$ ✓



b) $\boldsymbol{r}(t) = \begin{pmatrix} 7\sin t + 1 \\ 7\cos t \end{pmatrix}$

Starting position



$(x-1)^2 + y^2 = 49$ ✓

Circle centre, (1,0), and radius, 7, need to be accurate in picture.

c) Show algebraically how the vector equation in b) could be converted to the Cartesian equation.

$$x = 7\sin t + 1 \quad \Rightarrow \quad \sin t = \frac{x-1}{7}$$

$$y = 7\cos t \quad \Rightarrow \quad \cos t = \frac{y}{7}$$

✓

$$\sin^2 t + \cos^2 t = 1$$ ✓

$$\left(\frac{x-1}{7}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$$ ✓

$$(x-1)^2 + y^2 = 49$$

Question 9 **[3, 3, 2 = 8 marks]**

A particle P is projected from the origin with a speed of 60 ms^{-1} at an angle of 30° to the horizon. Assume that the only force acting on P is the gravitational force, 9.8 ms^{-2}.

a) Find an expression for the position vector of P t seconds after projection.

Initial velocity in component form: $\mathbf{v}(0) = < 60\cos 30, 60\sin 30 > = < 30\sqrt{3}, 30 >$.

Hence: $\mathbf{v}(t) = < 30\sqrt{3}, 30 - 9.8t >$

Integrate: $\mathbf{r}(t) = < 30\sqrt{3}\,t, 30t - 4.9t^2 >$ since $\mathbf{r}(0) = < 0, 0 >$.

b) Find the time taken for P to reach its maximum height and hence find the time of flight (the time the particle is in the air).

When P achieves maximum height, the vertical component of $\mathbf{v}(t)$ is zero.

Hence $30 - 9.8t = 0$

Thus $t = 3.06$ seconds

As the path is parabolic, it is symmetrical about the axis of symmetry.
Hence, the time taken for P to hit the ground again, T, is twice the time taken to reach the maximum height.

Therefore T = 6.12 seconds.

c) Find the horizontal displacement of P.

P hits the ground again after 6.12 seconds.
Substitute $t = 6.12$ into the horizontal component $\mathbf{r}(t)$:

$$r_x = (30\sqrt{3})(6.12) \approx 318 \text{ metres}$$